On Arithmetic Subgroups of Simple Algebraic Groups

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ABSTRACT

Arithmetic subgroups of simple isotropic algebraic groups are described as subgroups full of root elements.

STATEMENT OF RESULTS

Let k be a field, and G/k an absolutely almost simple algebraic group defined over k (see [5] for terminology and notation; an example to have in mind is $G = \mathrm{SL}_n$, $G_k = \mathrm{SL}_n(k)$). We assume everywhere in this note that the k-rank of G/k is at least 2. That is, every maximal k-split torus T in G/k has dimension $r \geqslant 2$. For each root of T we have the corresponding root subgroup of G/k. When G splits over k, every root subgroup is isomorphic to the additive group.

DEFINITION 1. A subgroup H of G_k is called *full* (of root elements) if every root subgroup with respect to every maximal k-split torus T contains a nontrivial element of H.

THEOREM 2. *Assume that G is either of classical type A, B, C, or D, but not of type ${}^3D_4$ or ${}^6D_4$, or split. Let H be a full subgroup of G_k. Then:*

(a) *the intersection of any finite family of conjugates of H is full;*
(b) *for any noncentral subgroup M of G_k normalized by H, $M \cap H$ is full.*

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Probably, the restriction on the type of G/k in the theorem is redundant. But the assumption $r \geqslant 2$ we made above is essential, as the example $G = \mathrm{SL}_2$ shows (see [7]).

A justification for Definition 1 comes from the fact that sometimes (maybe always) one can describe all the isomorphisms between full subgroups, and prove that every such an isomorphism (of abstract groups) can be extended to an algebraic isomorphism of the corresponding algebraic groups (see [3,4]). Another motivation is that a proof by Margulis of the arithmeticity of discrete subgroups in Lie groups obtains fullness as an intermediate step (see [1,2]).

Recall that a subgroup H of G_k is called *arithmetic* if k is a number field (that is, a finite algebraic extension of the field of rational numbers) and H is commensurable with G_K (that is, $H \cap G_K$ is of finite index in both groups), where K is the integers in k and G is a K-form of G/k (the definition, in fact, does not depend on the choice of K-form). More generally, S-arithmetic subgroups can be defined, where S is a set of places (valuations) of a global field k.

It is easy to see that every arithmetic (or S-arithmetic) subgroup is full. I believe that, conversely, every full subgroup contains an arithmetic subgroup (when k is a number field and $r \geqslant 2$), and I have checked this under the restrictions of Theorem 2:

THEOREM 3. *Under the conditions of Theorem 2, assume that k is a number field. Then a subgroup of G_k is full if and only if it contains an arithmetic subgroup.*

To state more general results, where k is an arbitrary field, we need more notation. Everywhere in this note, let K mean a subring with 1 of k such that k is its field of fractions, and I mean a nonzero ideal of K. For any maximal k-split torus T, any K-form G of G/k, and any I, let $E(T, G; I)$ denote the subgroup of $G_I = \ker(G_K \to G_{K/I})$ generated by the root elements with respect to T in G_I.

THEOREM 4. *Under the conditions of Theorem 2, for any K, G, T, I as above, $E(T, G; I)$ is a full subgroup of G_k.*

THEOREM 5. *Under the conditions of Theorem 2, in the case when G/k is of type C and* $\mathrm{char}(k) = 2$, *let us assume that the dimension of k over k^2 is either finite or countable. Let H be a full subgroup of G_k. Then there is a subring K of k (as above) such that for any K-form G of G/k and any T (as above) there exists I with $H \supset E(T, G; I)$.*

I believe that the restrictions on type of G/k in Theorem 2 can be removed also from Theorems 3, 4 and 5, but my knowledge of types ${}^3D_4$, ${}^6D_4$, and nonsplit exceptional types is not sufficient to do it. On the other hand, when G/k is of type C, $\mathrm{char}(k) = 2$, and the dimension of k over k^2 is uncountable, the conclusion of Theorem 5 is wrong for some full H; this is going to be shown in another publication.

PROOF OF THE THEOREMS FOR $G_k = \mathrm{SL}_n$

In this note, Theorems 2–5 will be proved for the simply connected G/k of type ${}^1A$, that is (see [5]), $G_k = \mathrm{SL}_n l$, the group of n by n matrices with the reduced norm 1 over a central division algebra l over k, $\dim_k l < \infty$. (The results for non-simply-connected G/k of this type follow easily from here; other types will be treated elsewhere.) Here a maximal k-split torus T consists of all diagonal matrices over k with determinant 1; any other maximal k-split torus is, as in the general case, a conjugate of T. So $r = k\text{-rank}(G/k) = n - 1 \geqslant 2$.

It is clear that every full (of root elements) subgroup of $G = \mathrm{SL}_n l$ is also a full (of transvections) subgroup of $\mathrm{GL}_n l$ in the sense of O'Meara [3] (see also [7]). It is proved in [7] that a subgroup H of $\mathrm{GL}_n l$ (where l is an arbitrary division ring and $n \geqslant 3$) is full of transvections if and only if $H \supset E_n B$ for a full subring B of l.

Here $E_n B$ is the subgroup generated by all elementary matrices $1_n + e_{i,j}(b)$ with $b \in B$ and $i \neq j$. A subset X of an associative ring R with 1 is called *full* if every element of R can be written as $u_1 u_2^{-1} = u_3^{-1} u_4$ with u_1, u_2, u_3, and u_4 in X and u_2, u_3 in $\mathrm{GL}_1 R$.

Therefore the following statement implies that a subgroup of $\mathrm{SL}_n l$ is full of root elements if and only if it is full of transvections:

(6) For any full subring B of l (where l is a finite-dimensional central division algebra over k), $B \cap k$ is full in k.

Results of [7] [together with (6)] give now Theorems 2, 4, and 5 for $G_k = \mathrm{SL}_n l$ ($n \geqslant 3$). To obtain Theorem 3, it remains to observe that when k is a number field, $E(G, T; I)$ is an arithmetic subgroup for any G, T, and I (see [6]; it suffices to check it for some G and T). ∎

Thus, we have only the statement (6) to prove.

Proof of (6). Let $z \neq 0$ be in k. We have to prove that $z = x_1 x_2^{-1}$ for some nonzero x_1 and x_2 in $k \cap B$. Since B is full in l, there are nonzero y_1 and y_2 in B such that $z = y_1 y_2^{-1}$.

It is well known (see, for example, [8, Chapter 9; 1, Proposition 3]) that every endomorphism p of the vector space l over k can be written as $pw = \sum_{i=1}^{m} b_i w c_i$ with b_i and c_i in l. We take p here to be a projection from l onto k. By [7, Lemma 4(a)], there is a nonzero d in B such that $b_i d$ and dc_i are in B for all i.

The vector subspace B' of l over k spanned by B is a full subring of l. Since every nonzero element of l satisfies a polynomial equation with coefficients in k and a nonzero constant term, every nonzero element of B' is invertible in B'. So $B' = l$, that is, B spans l over k.

Therefore we can find an element u in B outside $y_1^{-1} d^{-1}[\ker(p)] d^{-1}$. Then dy_1ud is in B and outside of $\ker(p)$; hence $0 \neq x_1 := p(dy_1ud) = \sum b_i dy_1udc_i \in B \cap k$. Set $x_2 := p(dy_2ud) \in B \cap k$. Since $zy_2 = y_1$, $z \in k$, and p is k-linear, we have $zx_2 = x_1$; hence $z = x_1 x_2^{-1}$. ∎

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